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G_2 generator matrix elements for degenerate representations in an $SU(3)$ basis*

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Abstract. Basis states and generator matrix elements are given for the degenerate representations $(a, 0)$ and $(0, b)$ of G_2 in an $SU(3)$ basis. We show that for any compact Lie group the elementary unwanted states, and hence incompatibilities between fundamental basis states in the character generator, are all of degree 2.

1. Introduction

In case the reader needs to be convinced that generator matrix elements are useful we quote a statement of Chacón and Moshinsky (1987) in a paper dealing with $Sp(6)$ and the nuclear symplectic model: 'One of the important problems in Lie algebras is to determine the matrix representation of their generators in a basis associated with a given irrep of the *corresponding group*.'

In this paper we develop the concept of basis states of irreducible representations for which the character generator of the group in question provides an integrity basis; usually the character generator is interpreted as yielding just the weights or characters of IRs. We propose that the states so defined be christened 'character states'. We believe them to be the most natural and simplest states for general use, in particular for computing generator matrix elements. Analogous states were used by Burdik *et al* (1992) for the degenerate representations $(0, 0, c)$ of $SO(7)$ in an $SU(2)^3$ basis (see also de Guise and Sharp 1991 for a case with a finite subgroup).

As a further example of their use we consider here the degenerate representations $(a, 0)$ and $(0, b)$ of $G_2 \supset SU(3)$, for which there is no missing label problem. We postpone for now the treatment of the generic representations (a, b) with a and b both non-zero.

Basis states and generator matrix elements for G_2 in an $SU(3)$ basis have been given by Sviridov, Smirnov and Tolstoy (1975). Their states are obtained by operating on the highest state of the G_2 representation with a product of powers of lowering short root generators and then projecting out the highest state of the $SU(3)$ IR whose highest state has that weight. Thus they obtain an overcomplete set of $SU(3)$ IRs. By restricting the exponents of the lowering generators they retain a complete non-redundant set; the superfluous states are expressed as linear combinations of the complete ones. When a generator is applied to a member of the complete set to obtain its matrix elements superfluous states are reached and must be expressed back in terms of the complete set. We believe our methods are superior and simpler (but 18 year later).

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The group–subgroup $G_2 \supset SU(3)$ does not seem to be used directly in physics. However the group chain $SO(7) \supset G_2 \supset SO(3)$ was introduced by Racah (1951) for the study of atomic electrons. We quote Judd (1968): 'The use of G_2 is an important feature of Racah's analysis and is largely responsible for the progress that has been made during the last decade in the analysis of actinide and rare-earth spectra.' For relevant generator matrix elements of $SO(7)$ one needs (according to the Wigner–Eckart theorem) only reduced matrix elements of a G_2 septet tensor and G_2 Clebsch–Gordan coefficients in an $SO(3)$ basis. The reduced matrix elements do not depend on internal G_2 labels and may be computed using the simplest G_2 basis states, say the character states for $G_2 \supset SU(3)$ described in this paper.

Section 2 deals with basis states for (a,0) and (0,b) representations. In section 3 the generator matrix elements are derived. Section 4 contains some concluding remarks. In the appendix, it is proved that elementary unwanted states for any compact Lie group, and hence incompatibilities in the character generator, are all of degree 2.

2. Basis states for degenerate representations of $G_2 \supset SU(3)$

The necessary ingredients for the calculation of $G_2 \supset SU(3)$ basis states and generator matrix elements between them are the $G_2 \supset SU(3)$ branching rules generating function and the character generator for G_2.

The generating function for $G_2 \supset SU(3)$ branching rules is (Gaskell *et al* 1978)

$$G(A, B; P, Q) = \frac{1}{(1-AP)(1-AQ)(1-BP)(1-BQ)}\left[\frac{1}{1-A} + \frac{BPQ}{1-BPQ}\right] \tag{2.1}$$

The dummy variables A, B carry the G_2 representation labels a, b as exponents and P, Q carry the $SU(3)$ representations labels p, q. The power series expansion

$$G = \sum_{a,b,p,q} A^a B^b P^p Q^q C_{abpq} \tag{2.2}$$

gives the multiplicity C_{abpq} of the $SU(3)$ representation (p, q) in the G_2 representation (a, b). Here $(1, 0)$ is the septet and $(0, 1)$ the 14-plet representation of G_2. But equation (2.1) does more than count $SU(3)$ multiplicities; we interpret $AP \sim \eta$, $AQ \sim \zeta^*$, $A \sim \theta$, $BP \sim \lambda$, $BQ \sim \nu^*$, $BPQ \sim \alpha$ as the highest states of the appropriate $SU(3)$ representations contained in the fundamental G_2 representations (see figure 1); the highest state of any $SU(3)$ representation contained in any G_2 representation is then given by the appropriate product of powers of them. Other $SU(3)$ states are obtained by applying $SU(3)$ lowering generators.

The basis states of the G_2 representation (a, b) are polynomials of degrees a, b in the states of the respective fundamental representations. That means that only stretched IRs

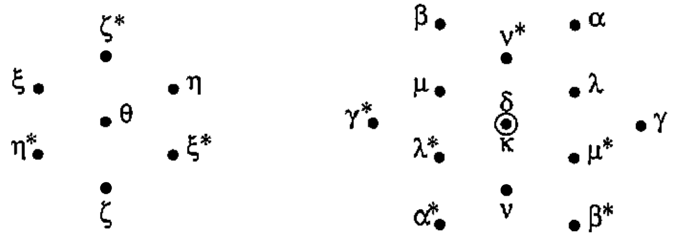

Figure 1. The basis states of the G_2 fundamental representations $(1, 0)$ and $(0, 1)$.

(representation labels additive) in the direct product of a copies of $(1,0)$ and b copies of $(0,1)$ are to be retained.

Consider the quadratic direct products

$$(1,0)^2_{28} = (2,0)_{27} + (0,0)_1 \tag{2.3a}$$

$$(0,1)^2_{105} = (0,2)_{77} + (2,0)_{27} + (0,0)_1 \tag{2.3b}$$

$$(1,0) \times (0,1)_{98} = (1,1)_{64} + (2,0)_{27} + (1,0)_7 . \tag{2.3c}$$

The square of a representation above means the symmetric (polynomial) part of the direct product of two copies. A subscript on a representation or product is its dimension. The stretched part of each product is the first representation on the right. The states of all the other representations are unwanted for the purpose of forming our polynomial basis.

To deal consistently with the unwanted states we need the help of the G_2 character generator (Gaskell 1983, Gaskell and Sharp 1981). With the character generator interpreted as providing an integrity basis for basis states we see that certain pairs of fundamental representation basis states never appear multiplied; each such incompatible pair appears as a term in exactly one unwanted state (or more than one appear in the same number of unwanted states all of the same weight). Equating all the unwanted states to zero enables us to solve for each incompatible pair in terms of pairs that are compatible according to the character generator. Whenever an incompatible pair appears as a result of applying a generator to another state we eliminate it by means of these incompatibility equations.

It is shown in the appendix that the elementary unwanted IRs are all of degree 2, a fact verified straightforwardly in the case of G_2.

We remark that although our states correspond one-to-one to all states of all G_2 representations they are still contaminated by unwanted states belonging to lower representations; that does not matter for the purpose of computing generator matrix elements.

We deal first with $(a,0)$ basis states. The relevant $G_2 \supset SU(3)$ branching rules generating function (see equation (2.1)) is

$$G(A,0;P,Q) = \frac{1}{(1-A)(1-AP)(1-AQ)} \tag{2.4}$$

from which we conclude that the highest state of the $SU(3)$ representation (p,q) in the G_2 representation $(a,0)$ is (we suppress the G_2 labels)

$$\left| \begin{matrix} p\ q \\ p\ p\ (p+2q)/3 \end{matrix} \right\rangle = N_{pq}\eta^p\zeta^{*q}\theta^{a-p-q} \tag{2.5}$$

the internal $SU(3)$ labels are respectively t, m, y, with the isospin labels doubled to avoid half-odd values. The normalization constant N_{pq} will be determined in section 3. Other states are determined by applying lowering $SU(3)$ generators to (2.5). In differential form, suitable for operating on (2.5), they are

$$\begin{aligned} E_{21} &= \xi\partial_\eta - \eta^*\partial_{\xi^*} \\ E_{32} &= \zeta\partial_\xi - \xi^*\partial_{\zeta^*} \\ E_{31} &= \zeta\partial_\eta - \eta^*\partial_{\zeta^*} . \end{aligned} \tag{2.6}$$

The branching rules implied by (2.5) are $0 \leqslant p+q \leqslant a$.

The character generator for $(a,0)$ representations is (Gaskell and Sharp 1981)

$$\frac{1}{\eta\zeta\eta^*\zeta^*\theta}\left[\frac{1}{\xi} + \frac{\xi^*}{\xi^*}\right] \tag{2.7}$$

where we have adopted the space-saving convention that a variable in the denominator stands for unity minus that variable. In (2.7) the variables may be regarded as representing the seven states of the fundamental G_2 representation $(1, 0)$ (see figure 1), or alternatively, as products of dummy variables carrying the G_2 representation labels and weight components as exponents. Then we have

$$\begin{array}{llll} \eta = A\sigma^{-1}\tau & \xi = A\sigma^{2}\tau^{-1} & \zeta = A\sigma^{-1} & \\ \eta^* = A\sigma\tau^{-1} & \xi^* = A\sigma^{-2}\tau & \zeta^* = A\sigma & \theta = A \end{array} \tag{2.8}$$

where σ, τ carry the G_2 weight components in a fundamental weights basis. According to (2.7) there is just one incompatible pair, $\xi\xi^*$.

Also, according to (2.3*a*) there is just one unwanted state, the G_2 scalar on the right-hand side. It is

$$M = \eta\eta^* + \xi\xi^* + \zeta\zeta^* + \tfrac{1}{2}\theta^2 . \tag{2.9}$$

Accordingly we eliminate $\xi\xi^*$ whenever it arises by the replacement

$$\xi\xi^* \to -\eta\eta^* - \zeta\zeta^* - \tfrac{1}{2}\theta^2 . \tag{2.10}$$

For $(0, b)$ representations the character generator of G_2 is (Gaskell and Sharp 1981)

$$\begin{aligned} \frac{1}{\nu\nu^*\delta}\Bigg[& \frac{1}{\alpha^*\gamma^*\kappa} + \frac{\beta}{\beta\gamma^*\kappa} + \frac{\alpha}{\alpha\beta\kappa} + \frac{\beta^*}{\alpha^*\beta^*\kappa} + \frac{\gamma}{\beta^*\gamma\kappa} + \frac{\alpha\gamma}{\alpha\gamma\kappa} \\ & + \frac{\lambda^*}{\alpha^*\gamma^*\lambda^*} + \frac{\mu^*}{\alpha^*\beta^*\mu^*} + \frac{\lambda^*\mu^*}{\alpha^*\lambda^*\mu^*} + \frac{\gamma\mu^*}{\beta^*\gamma\mu^*} + \frac{\mu}{\beta\gamma^*\mu} + \frac{\alpha\mu}{\alpha\beta\mu} \\ & + \frac{\lambda}{\alpha\lambda\mu} + \frac{\gamma\lambda}{\alpha\gamma\lambda} + \frac{\lambda^*\mu}{\gamma^*\lambda^*\mu} + \frac{\lambda\lambda^*}{\lambda\lambda^*\mu} + \frac{\lambda\mu^*}{\lambda\lambda^*\mu^*} + \frac{\gamma\lambda\mu^*}{\gamma\lambda\mu^*}\Bigg] . \end{aligned} \tag{2.11}$$

The variables in (2.11) are the 14 basis states of the fundamental representation $(0, 1)$ (see figure 1). Again a variable in the denominator means one minus that variable. Each variable can also be interpreted as a product of powers of dummy variables

$$\begin{array}{llll} \alpha = B\tau & \alpha^* = B\tau^{-1} & \beta = B\sigma^3\tau^{-1} & \beta^* = B\sigma^{-3}\tau \\ \gamma = B\sigma^{-3}\tau^2 & \gamma^* = B\sigma^3\tau^{-2} & \delta = BT^2 & \kappa = B \\ \lambda = B\sigma^{-1}\tau & \lambda^* = B\sigma\tau^{-1} & \mu = B\sigma^2\tau^{-1} & \mu^* = B\sigma^{-2}\tau \\ \nu = B\sigma^{-1} & \nu^* = B\sigma . & & \end{array} \tag{2.12}$$

In deriving equation (2.11), we followed Gaskell and Sharp (1981) in substituting the $SU(3)$ character generator into the $G_2 \supset SU(3)$ branching rules generating function; but in the $SU(3)$ character we kept a dummy T carrying as its exponent the $SU(2)$ (isospin) representation label. In (2.12) we retained this T dependence only for δ and κ as a means of distinguishing them; thus δ is the $m = 0$ member of an $SU(2)$ triplet, while κ is an $SU(2)$ singlet. The compatibility table, obtained by examination of (2.11), is shown in figure 2. The 28 incompatible pairs correspond to the unwanted states of the representations $(2, 0)$ (dimension 27) and $(0, 0)$ (dimension 1) on the right-hand side of equation (2.3*b*).

We have determined the 28 unwanted states, set each of them equal to zero, and solved for each incompatible pair. We give seven of the resulting replacements; ones actually

	α^*	β	β^*	γ	γ^*	κ	λ	λ^*	μ	μ^*
α	×		×		×			×		×
α^*		×		×			×		×	
β			×	×			×	×		×
β^*					×		×	×	×	
γ					×			×	×	
γ^*							×			×
κ							×	×	×	×
λ										
λ^*										
μ										×

Figure 2. Incompatibility table. Each incompatible pair of (0, 1) states is marked with a cross.

needed in section 3

$$\begin{aligned}
\alpha\lambda^* &\to \frac{2^{1/2}}{2}\delta\nu^* - \frac{6^{1/2}}{6}\kappa\nu^* - \frac{3^{1/2}}{3}\lambda\mu \\
\alpha\mu^* &\to \frac{3^{1/2}}{3}\lambda^2 - \gamma\nu^* \\
\beta\gamma &\to \frac{6^{1/2}}{6}\alpha\kappa + \frac{2^{1/2}}{2}\alpha\delta + \frac{1}{3}\lambda\nu^* \\
\beta\lambda &\to \frac{3^{1/2}}{3}\nu^{*2} + \alpha\mu \\
\gamma\mu &\to \frac{1}{2}\alpha\nu + \frac{2^{1/2}}{2}\delta\lambda + \frac{3^{1/2}}{6}\mu^*\nu^* \\
\kappa\lambda &\to \frac{6^{1/2}}{2}\alpha\nu - \frac{2^{1/2}}{2}\mu^*\nu^* .
\end{aligned} \tag{2.13}$$

These substitutions are to be made whenever an incompatible pair arises during our computations.

The $G_2 \supset SU(3)$ branching rules generating function for $(0, b)$ representations is (see (2.1))

$$G(0, B; P, Q) = \frac{1}{(1 - BP)(1 - BQ)(1 - BPQ)} \tag{2.14}$$

from which we see that the highest state of the $SU(3)$ representation (p, q) is

$$\left| \begin{matrix} p\ q \\ p\ p\ (p+2q)/3 \end{matrix} \right\rangle = N_{pq}\lambda^{b-q}\nu^{*b-p}\alpha^{p+q-b} \tag{2.15}$$

We have suppressed the G_2 representation label. The normalization constant N_{pq} will be determined in section 3. The branching rules implied by (2.15) are $p + q \geqslant b \geqslant p, q$. States other than the highest of $SU(3)$ representations are obtained by applying lowering

$SU(3)$ generators to (2.15). In differential form they are

$$
\begin{aligned}
E_{21} &= \beta\partial_\alpha - \alpha^*\partial_{\beta^*} + \mu\partial_\lambda - \lambda^*\partial_{\mu^*} + 2^{1/2}(\delta\partial_\gamma - \gamma^*\partial_\delta) \\
E_{32} &= \gamma\partial_\alpha - \alpha^*\partial_{\gamma^*} + \nu\partial_\mu - \mu^*\partial_{\nu^*} + \frac{6^{1/2}}{2}(\kappa\partial_\beta - \beta^*\partial_\kappa) + \frac{2^{1/2}}{2}(\delta\partial_\beta - \beta^*\partial_\delta) \\
E_{31} &= \gamma^*\partial_\beta - \beta^*\partial_\gamma + \nu\partial_\lambda - \lambda^*\partial_{\nu^*} + \frac{6^{1/2}}{2}(\kappa\partial_\alpha - \alpha^*\partial_\kappa) + \frac{2^{1/2}}{2}(\alpha^*\partial_\delta - \delta\partial_\alpha)\,.
\end{aligned} \tag{2.16}
$$

3. The generator matrix elements

We now calculate generator matrix elements with respect to the $(a,0)$ basis states (2.5) and the $(0,b)$ basis states (2.15) and, incidentally, determine the normalization constants N_{pq} in both cases.

The matrix elements of the eight $SU(3)$ generators are well known (Gel'fand and Zetlin 1950); the other six G_2 generators form two $SU(3)$ tensors $G^{(10)}$ and $G^{(01)}$ which transform by the indicated $SU(3)$ representations. According to the $SU(3)$ Wigner–Eckart theorem the matrix elements of $G^{(10)}$ are given in terms of its reduced matrix elements by

$$
\left\langle \begin{matrix} p_2\, q_2 \\ t_2\, m_2\, y_2 \end{matrix} \middle| G^{(10)}_{t,m,y} \middle| \begin{matrix} p_1\, q_1 \\ t_1\, m_1\, y_1 \end{matrix} \right\rangle = \langle\, p_2\, q_2 \,\|\, G^{(10)} \,\|\, p_1\, q_1 \,\rangle \left\langle \begin{matrix} p_1\, q_1 \\ t_1\, m_1\, y_1 \end{matrix} ; \begin{matrix} 1\,0 \\ t\; m\; y \end{matrix} \middle| \begin{matrix} p_2\, q_2 \\ t_2\, m_2\, y_2 \end{matrix} \right\rangle
$$
$$
\times \left[(p_2+1)(q_2+1)(p_2+q_2+2)/2\right]^{-1/2}\,. \tag{3.1}
$$

Here

$$
\left\langle \begin{matrix} p_1\, q_1 \\ t_1\, m_1\, y_1 \end{matrix} ; \begin{matrix} 1\,0 \\ t\; m\; y \end{matrix} \middle| \begin{matrix} p_2\, q_2 \\ t_2\, m_2\, y_2 \end{matrix} \right\rangle
$$

is an $SU(3)$ Clebsch–Gordan coefficient (for a formula see Moshinsky (1962), Resnikoff (1967) or Asherova and Smirnov (1968)). Thus it is necessary for us to give only the reduced matrix elements $\langle\, p_2\, q_2 \,\|\, G^{(10)} \,\|\, p_1\, q_1 \,\rangle$ where (p_2, q_2) is (p_1+1, q_1), (p_1-1, q_1+1) or (p_1, q_1-1). A formula similar to (3.1) exists for matrix elements of $G^{(01)}$. Because $G^{(10)}$ and $G^{(01)}$ are Hermitian conjugate tensors ($G^{(10)\dagger}_{0,0,-2/3} = G^{(01)}_{0,0,2/3}$, $G^{(10)\dagger}_{1,1,1/3} = G^{(01)}_{1,-1,-1/3}$, $-G^{(10)\dagger}_{1,-1,1/3} = G^{(01)}_{1,1,-1/3}$), $\langle\, p_1\, q_1 \,\|\, G^{(01)} \,\|\, p_2\, q_2 \,\rangle$ is equal to $\langle\, p_2\, q_2 \,\|\, G^{(01)} \,\|\, p_1\, q_1 \,\rangle$ within a sign.

We deal first with $(a,0)$ generator matrix elements. In differential form we have

$$
G^{(01)}_{0,0,2/3} = \eta\partial_{\xi^*} - \xi\partial_{\eta^*} + 2^{1/2}(\theta\partial_\zeta - \zeta^*\partial_\theta). \tag{3.2}
$$

Applying it to the state (2.5) we find immediately that

$$
\left\langle \begin{matrix} p\;\; q+1 \\ p\;\, p\;\, (p+2q+2)/3 \end{matrix} \middle| G^{(01)}_{0,0,2/3} \middle| \begin{matrix} p\;\; q \\ p\;\, p\;\, (p+2q)/3 \end{matrix} \right\rangle = -2^{1/2}(a-p-q)\frac{N_{p,q}}{N_{p,q+1}}. \tag{3.3}
$$

Similarly, applying $G^{(10)}_{0,0,-2/3}$ to $\left| \begin{matrix} p\; q+1 \\ p\; p\; (p+2q+2)/3 \end{matrix} \right\rangle$ (getting $G^{(10)}_{0,0,-2/3}$ from $G^{(01)}_{0,0,2/3}$ by interchanging variables and differentiations ($\eta \leftrightarrow \partial_\eta$, etc)), we find, using (2.10)

$$
\left\langle \begin{matrix} p\;\; q \\ p\;\, p\;\, (p+2q)/3 \end{matrix} \middle| G^{(10)}_{0,0,-2/3} \middle| \begin{matrix} p\;\; q+1 \\ p\;\, p\;\, (p+2q+2)/3 \end{matrix} \right\rangle
$$
$$
= -\frac{2^{1/2}(q+1)(a+p+q+5)}{2(p+q+3)}\frac{N_{p,q+1}}{N_{p,q}} \tag{3.4a}
$$

$$\left\langle \begin{matrix} p+1\ q+1 \\ p\ p\ (p+2q)/3 \end{matrix} \middle| G^{(10)}_{0,0,-2/3} \middle| \begin{matrix} p\ q+1 \\ p\ p\ (p+2q+2)/3 \end{matrix} \right\rangle$$

$$= (a-p-q-1)\left[\frac{2(p+2)}{(p+1)(p+q+3)}\right]^{1/2} \frac{N_{p,q+1}}{N_{p+1,q+1}} \tag{3.4b}$$

$$\left\langle \begin{matrix} p-1\ q+2 \\ p\ p\ (p+2q)/3 \end{matrix} \middle| G^{(10)}_{0,0,-2/3} \middle| \begin{matrix} p\ q+1 \\ p\ p\ (p+2q+2)/3 \end{matrix} \right\rangle = -\frac{p}{(q+2)^{1/2}} \frac{N_{p,q+1}}{N_{p-1,q+2}}. \tag{3.4c}$$

Equating the matrix elements in (3.3) and (3.4a) gives a simple recurrence relation for $N_{p,q}$

$$\frac{N_{p,q+1}}{N_{p,q}} = \left[\frac{2(p+q+3)(a-p-q)}{(q+1)(a+p+q+5)}\right]^{1/2} \tag{3.5}$$

with the initial condition

$$N_{p,a-p} = [p!(a-p)!]^{-1/2}. \tag{3.6}$$

(The state $\left| \begin{smallmatrix} p\ a-p \\ p\ p\ (2a-p)/3 \end{smallmatrix} \right\rangle = N_{p,a-p}\eta^p \zeta^{*a-p}$ contains no unwanted states and is easily normalized.) We find

$$N_{p,q} = \left[\frac{(2a+4)!(p+q+2)!}{2^{a-p-q}(a+2)!p!q!(a-p-q)!(a+p+q+4)!}\right]^{1/2}. \tag{3.7}$$

$N_{p,q}$ normalizes the wanted part of the state (2.5), which, incidentally, has not been isolated.

With $N_{p,q}$ given by (3.7) we have explicit expressions for the matrix elements in (3.4a) and for the corresponding ones involving $G^{(01)}_{0,0,2/3}$. For the desired reduced matrix elements we find

$$\langle p\ q-1 \,\|\, G^{(10)} \,\|\, p\ q \rangle = \langle p\ q \,\|\, G^{(01)} \,\|\, p\ q-1 \rangle$$
$$= -[(p+1)q(q+1)(a-p-q+1)(a+p+q+4)/2]^{1/2} \tag{3.8a}$$

$$\langle p+1\ q \,\|\, G^{(10)} \,\|\, p\ q \rangle = \langle p\ q \,\|\, G^{(01)} \,\|\, p+1\ q \rangle$$
$$= [(p+1)(p+2)(q+1)(a-p-q)(a+p+q+5)/2]^{1/2} \tag{3.8b}$$

$$\langle p-1\ q+1 \,\|\, G^{(10)} \,\|\, p\ q \rangle = -\langle p\ q \,\|\, G^{(01)} \,\|\, p-1\ q+1 \rangle$$
$$= -[p(p+1)(q+1)(q+2)(p+q+2)/2]^{1/2}. \tag{3.8c}$$

We turn to the case of $(0, b)$ generator matrix elements, following the same procedure as for $(a, 0)$. In differential form we have

$$G^{(01)}_{0,0,2/3} = 2^{1/2}\nu^*\partial_\kappa - 2^{1/2}\kappa\partial_\nu - 3^{1/2}\alpha\partial_\lambda + 3^{1/2}\lambda^*\partial_{\alpha^*} - 2\lambda\partial_{\mu*} + 2\mu\partial_{\lambda^*}$$
$$+3^{1/2}\mu^*\partial_{\beta^*} - 3^{1/2}\beta\partial_\mu. \tag{3.9}$$

Applying it to the state (2.15) yields immediately

$$\left\langle \begin{matrix} p\ q+1 \\ p\ p\ (p+2q+2)/3 \end{matrix} \middle| G^{(01)}_{0,0,2/3} \middle| \begin{matrix} p\ q \\ p\ p\ (p+2q)/3 \end{matrix} \right\rangle = -3^{1/2}(b-q)\frac{N_{p,q}}{N_{p,q+1}}. \tag{3.10}$$

The application of $G^{(10)}_{0,0,-2/3}$ to $\left|\begin{smallmatrix} p\ p+1 \\ p\ p\ (p+2q+2)/3 \end{smallmatrix}\right\rangle$ is algebraically more complicated than for the $(a,0)$ case, and we needed Mathematica to help us implement it. We find (using (2.13))

$$\left\langle \begin{matrix} p\ q \\ p\ p\ (p+2q)/3 \end{matrix} \middle| G^{(10)}_{0,0,-2/3} \middle| \begin{matrix} p\ q+1 \\ p\ p\ (p+2q+2)/3 \end{matrix} \right\rangle$$

$$= -\frac{(b+q+3)(b+p+q+4)(p+q-b+1)}{3^{1/2}(q+2)(p+q+3)} \frac{N_{p,q+1}}{N_{p,q}} \tag{3.11a}$$

$$\left\langle \begin{matrix} p+1\ q+1 \\ p\ p\ (p+2q)/3 \end{matrix} \middle| G^{(10)}_{0,0,-2/3} \middle| \begin{matrix} p\ q+1 \\ p\ p\ (p+2q+2)/3 \end{matrix} \right\rangle$$

$$= (b-p)\left[\frac{3(p+2)}{(p+1)(p+q+3)}\right]^{1/2} \frac{N_{p,q+1}}{N_{p+1,q+1}} \tag{3.11b}$$

$$\left\langle \begin{matrix} p-1\ q+2 \\ p\ p\ (p+2q)/3 \end{matrix} \middle| G^{(10)}_{0,0,-2/3} \middle| \begin{matrix} p\ q+1 \\ p\ p\ (p+2q+2)/3 \end{matrix} \right\rangle$$

$$= \frac{(b-q-1)(b+p+2)}{(p+1)(q+2)^{1/2}} \frac{N_{p,q+1}}{N_{p-1,q+2}}. \tag{3.11c}$$

Equating the matrix elements in (3.10) and (3.11a) gives a recurrence relation for $N_{p,q}$

$$\frac{N_{p,q+1}}{N_{p,q}} = \left[\frac{3(b-q)(q+2)(p+q+3)}{(b+q+3)(b+p+q+4)(p+q-b+1)}\right]^{1/2} \tag{3.12}$$

which, iterated out to $q = b$ (q's largest value according to (2.15)), yields

$$N_{p,q} = N_{p,b}\frac{1}{(3^{b-q})^{1/2}}\left[\frac{(2b+2)!p!(q+1)!(2b+p+3)!(p+q+2)!}{(b+1)!(b+p+2)!(b+q+2)!(b-q)!(p+q-b)!}\right]^{1/2}$$

$$\times\frac{1}{\left[(b+p+q+3)!\right]^{1/2}}. \tag{3.13}$$

$\left|\begin{smallmatrix} p\ b \\ p\ p\ (p+2b)/3 \end{smallmatrix}\right\rangle = N_{p,b}\nu^{*b-p}\alpha^p$ is still contaminated with unwanted states. To normalize it we equate

$$\left\langle \begin{matrix} p+1\ b \\ p+1\ p+1\ (p+2b+1)/3 \end{matrix} \middle| G^{(10)}_{1,1,1/3} \middle| \begin{matrix} p\ b \\ p\ p\ (p+2b)/3 \end{matrix} \right\rangle$$

$$= 3^{1/2}(b-p)\frac{N_{p,b}}{N_{p+1,b}}$$

and

$$\left\langle \begin{matrix} p\ b \\ p\ p\ (p+2b)/3 \end{matrix} \middle| G^{(01)}_{1,-1,-1/3} \middle| \begin{matrix} p+1\ b \\ p+1\ p+1\ (p+2b+1)/3 \end{matrix} \right\rangle$$

$$= \frac{3^{1/2}}{3}\frac{(p+1)(2b+p+4)}{p+2}\frac{N_{p+1,b}}{N_{p,b}}$$

and get a recurrence relation for $N_{p,b}$

$$\frac{N_{p+1,b}}{N_{p,b}} = \left[\frac{3(b-p)(p+2)}{(p+1)(2b+p+4)}\right]^{1/2} \tag{3.14}$$

with the initial condition

$$N_{b,b} = b!^{-1/2}. \tag{3.15}$$

(The state $\left| \begin{smallmatrix} b\ b \\ b\ b\ b \end{smallmatrix} \right\rangle = N_{b,b}\alpha^b$ contains nothing unwanted and is easily normalized). The solution is

$$N_{p,b} = \left[\frac{(3b+3)!(p+1)}{3^{b-p}(b+1)!(b-p)!(2b+p+3)!} \right]^{1/2} \tag{3.16}$$

and from (3.13) we get

$$N_{p,q} = \frac{1}{(3^{2b-p-q})^{1/2}(b+1)!} \left[\frac{(2b+2)!(3b+3)!(p+1)!(q+1)!(p+q+2)!}{(b-p)!(b-q)!(b+p+2)!(b+q+2)!} \right]^{1/2}$$
$$\times \left[\frac{1}{(p+q-b)!(b+p+q+3)!} \right]^{1/2}. \tag{3.17}$$

As a check on (3.17) we normalize explicitly the wanted part of the state

$$\left| \begin{matrix} 1\ 1 \\ 1\ 1\ 1 \end{matrix} \right\rangle = N_{1,1}\lambda\nu^* \tag{3.18}$$

belonging to the G_2 representation $(0, 2)$. The wanted part of (3.18) is (subtract the multiple of the unwanted state

$$\lambda\nu^* + \frac{6^{1/2}}{2}\alpha\kappa + \frac{3(2^{1/2})}{2}\alpha\delta - 3\beta\gamma$$

that leaves the result orthogonal to the unwanted state)

$$N_{1,1}\left[\frac{15}{16}\lambda\nu^* - \frac{6^{1/2}}{32}\alpha\kappa - \frac{3(2^{1/2})}{32}\alpha\delta + \frac{3}{16}\beta\gamma \right] \tag{3.19}$$

from which we see $N_{1,1} = (\frac{16}{15})^{1/2}$, in agreement with $N_{p,q}$, equation (3.17) with $p = q = 1$, $b = 2$.

With $N_{p,q}$ given by (3.17) we have explicit expressions for the matrix elements in (3.11*a*) and for the corresponding ones involving $G^{(01)}_{0,0,2/3}$. We find for the desired reduced matrix elements

$$\langle p\ q-1 \,\|\, G^{(10)} \,\|\, p\ q \rangle = \langle p\ q \,\|\, G^{(01)} \,\|\, p\ q-1 \rangle$$
$$= -\left[(p+1)(b-q+1)(b+q+2)(b+p+q+3)(p+q-b)/2\right]^{1/2} \tag{3.20a}$$

$$\langle p+1\ q \,\|\, G^{(10)} \,\|\, p\ q \rangle = \langle p\ q \,\|\, G^{(01)} \,\|\, p+1\ q \rangle$$
$$= \left[(b-p)(q+1)(b+p+3)(p+q-b+1)(b+p+q+4)/2\right]^{1/2} \tag{3.20b}$$

$$\langle p-1\ q+1 \,\|\, G^{(10)} \,\|\, p\ q \rangle = -\langle p\ q \,\|\, G^{(01)} \,\|\, p-1\ q+1 \rangle$$
$$= \left[(b-p+1)(b-q)(b+p+2)(b+q+3)(p+q+3)/2\right]^{1/2}. \tag{3.20c}$$

4. Concluding remarks

We have defined the matrix elements of a generator G between states $|a\rangle$ and $|b\rangle$ of a complete non-redundant set as $(b|G|a)$ = coefficient of $|b\rangle$ in $G|a\rangle$. As long as the states are orthonormal this is equivalent to the usual definition $\langle b|G|a\rangle$ = overlap of $|b\rangle$ with $G|a\rangle$; this is the case when the subgroup used to define the states provides a complete set of labels, as for the states we deal with in this article.

In a future paper we hope to treat generic representations of G_2 in an $SU(3)$ basis; there is then a missing label. There is no real need to orthonormalize the states. The matrices obtained can be multiplied in the usual way and functions of them, as for example an energy operator in the enveloping algebra, can be diagonalized in the usual way.

The basis states are defined as the 'wanted' parts of products of powers of mutually compatible sets of basis states of the fundamental representations. Mutually compatible means they all appear in the same term of the character generator. As will be shown in the appendix, the elementary incompatibilities are between pairs only of states, or, equivalently, elementary unwanted factors represent states that are quadratic in the basis states of the fundamental representations. When the incompatibilities are known the complete character generator can be written down straightforwardly. It will be interesting to see whether a consistent set of incompatibilities and hence the complete character generator for a group can be obtained by looking only at representations with two labels at a time that is non-zero.

Appendix. The degrees of elementary unwanted states

Polynomials in the states of the fundamental representations of a simple Lie group G can be decomposed according to irreducible representations of G. According to Cartan (1894) the stretched (representation labels additive) part of the product of λ_i copies of the ith fundamental representation, $i = 1, \ldots, l$, provides basis states for the IR $(\lambda_1, \lambda_2, \ldots \lambda_l)$; we call such states *wanted* and all other (unstretched) states *unwanted*. An elementary unwanted state is one which does not contain as a factor an unwanted state of lower degree. It is the purpose of this appendix to show that all elementary unwanted states are of degree 2.

Consider a state belonging to the IR λ of G which is contained in the direct product of the n IRs $\lambda_1, \ldots, \lambda_n$. In application $\lambda_1, \ldots, \lambda_n$ will be fundamental IRs but for now they are not necessarily so. Apply to our state the second order Casimir operator C_2 of G. Since it is of second degree in the generators of G it is a sum of parts which depend on the individual λ_i $(i = 1, \ldots, n)$ and parts which depend on pairs λ_i, λ_j $(i > j = 1, \ldots, n)$.

According to Racah (1965) the eigenvalue of C_2 acting on a state of the IR λ is

$$(M_\lambda + R)^2 - R^2 = M_\lambda^2 + 2M_\lambda \cdot R$$

where M_λ is the highest weight of λ and R is the sum of the fundamental weights. Hence the part of C_2 which depends on the individual IRs λ_i is

$$\sum_{i=1}^{n} \left(M_{\lambda_i}^2 + 2M_{\lambda_i} \cdot R\right) .$$

We assume that the pairs λ_i, λ_j are coupled in a stretched way; otherwise even if our state is unwanted it would not be elementary. So the part of C_2 which acts on λ_i and λ_j yields

$$\left(M_{\lambda_i} + M_{\lambda_j}\right)^2 + 2\left(M_{\lambda_i} + M_{\lambda_j}\right) \cdot R$$

and the part depending specifically on the pair is $2M_{\lambda_i} \cdot M_{\lambda_j}$. The complete eigenvalue of C_2 is

$$\sum_{i=1}^{n} \left(M_{\lambda_i}^2 + 2M_{\lambda_i} \cdot R\right) + 2 \sum_{i>j=1}^{n} M_{\lambda_i} \cdot M_{\lambda_j} = \left(\sum_{i=1}^{n} M_{\lambda_i}\right)^2 + 2\sum_{i=1}^{n} M_{\lambda_i} \cdot R$$

as if the coupling of $\lambda_1, \ldots, \lambda_n$ is completely stretched. This shows that the coupling is stretched, for any non-stretched coupling λ would have M_λ closer to the origin of weight space and hence a smaller eigenvalue of C_2.

We complete the appendix by remarking that since the product of an incompatible set of fundamental basis states is one term in the expression for an unwanted state, the elementary incompatibilities are all between pairs of fundamental basis states (or one such state may be incompatible with itself).

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